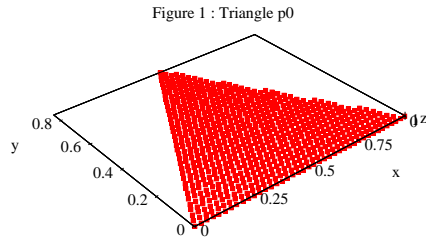


1 What's this ?

This is a memo to remind me that how a program sankaku makes tetrahedron globe from cylindrical projection map. The program is available at my homepage <http://www.2f.biglobe.ne.jp/~notchi/>.

2 Defining Base Triangle

Define a right triangle has a side length 1 and a bottom side lies (0,0,0) to (1,0,0). Let this triangle be p_0 .



3 Constructing Four Triangles

3.1 Triangle p_1

Transfer the triangle to p_1 . p_1 is obtained following matrix calculation.

- Rotate it $\pi + \cos^{-1} \frac{1}{3}$ radians around x axis.

$$p'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi + \cos^{-1} \frac{1}{3}) & \sin(\pi + \cos^{-1} \frac{1}{3}) \\ 0 & -\sin(\pi + \cos^{-1} \frac{1}{3}) & \cos(\pi + \cos^{-1} \frac{1}{3}) \end{pmatrix} p_0$$

- Rotate it $\frac{\pi}{6}$ radians around z axis

$$p''_1 = p'_1 \begin{pmatrix} \cos \frac{2\pi}{3} & \sin \frac{2\pi}{3} & 0 \\ -\sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

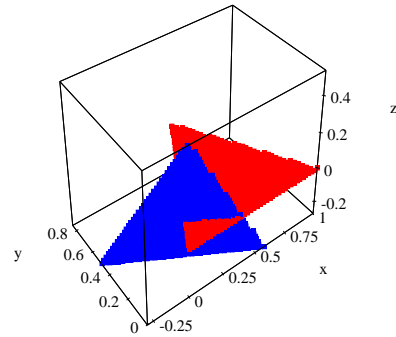
- Move it this much

$$p_1^{(3)} = p''_1 + \begin{pmatrix} -f \sin \frac{\pi}{6} \\ f \cos \frac{\pi}{6} \\ 0 \end{pmatrix} \text{ where } f = \frac{1}{2} \cos \frac{\pi}{6}$$

- Move it again this much

$$p_1 = p_1^{(3)} + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{6}}{3} \end{pmatrix}$$

Figure 2 : Triangle p_0 (red) and p_2 (blue)

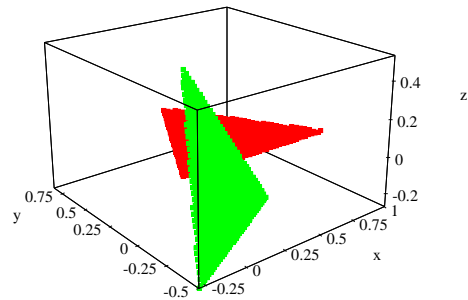


3.2 Triangle p_2

To obtain p_2 , p_1 is rotated and moved by following Matrix Calculation p_2 is Rotate p_1 $\frac{2\pi}{3}$ radians around z axis. In other words,

$$p_2 = \begin{pmatrix} \cos \frac{2\pi}{3} & \sin \frac{2\pi}{3} & 0 \\ -\sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} p_1$$

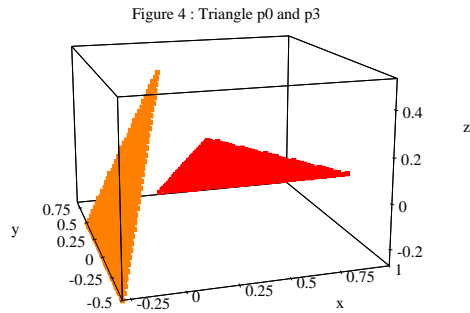
Figure 3 : Triangle p_0 (red) and p_2 (green)



3.3 Triangle p_3

Similarly to p_2 , Triangle p_3 is obtained by rotating p_1 $-\frac{2\pi}{3}$ radians around z axis. In other words,

$$p_3 = \begin{pmatrix} \cos \frac{-2\pi}{3} & \sin \frac{-2\pi}{3} & 0 \\ -\sin \frac{-2\pi}{3} & \cos \frac{-2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} p_1$$



3.4 Triangle p_4

Triangle p_4 is obtained by following rotation and movement of p_0 .

- Rotate $\frac{-\pi}{6}$ radians around z axis

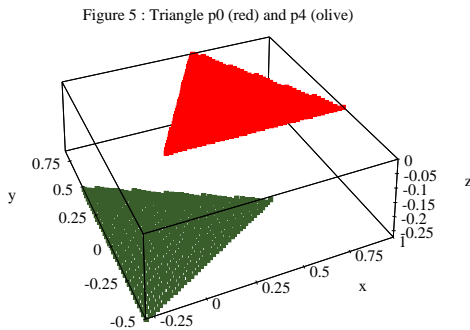
$$p'_4 = \begin{pmatrix} \cos \frac{-\pi}{6} & \sin \frac{-\pi}{6} & 0 \\ -\sin \frac{-\pi}{6} & \cos \frac{-\pi}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} p_0$$

- Move it this much

$$p''_4 = p'_4 + \begin{pmatrix} \frac{\sqrt{3}}{6} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$

- Move it $\frac{-\sqrt{6}}{3}$ along z axis.

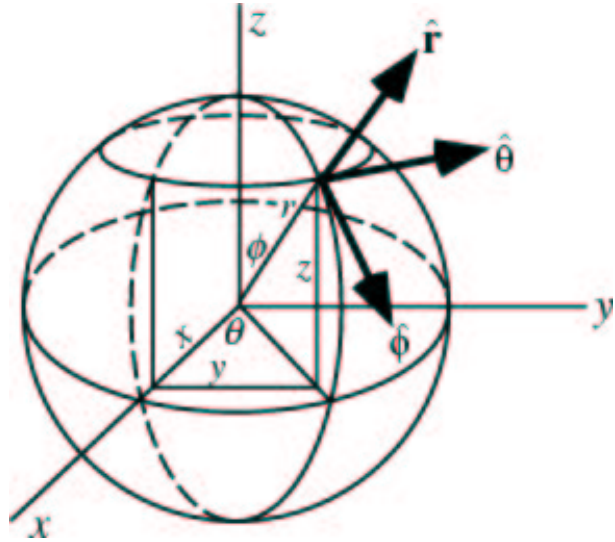
$$p_4 = p''_4 + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{6}}{3} \end{pmatrix}$$



4 Spherical Coordinate

We've got right tetrahedron with a side length 1 and origin centered. Now we project it to sphere with radius 1 by using

spherical coordinate. Refere to figure 6 for variable θ and ϕ .



- Defining unit vector $\hat{x} < 1, 0, 0 >$.
- Projection of vector $< x, y, z >$ onto $x - y$ plane is r_{xy} . So

$$r_{xy} = < x, y, 0 >$$

- Angle θ is obtained by dot product

$$\begin{aligned} \hat{x} \cdot r_{xy} &= |\hat{x}| |r_{xy}| \cos \theta \\ \theta &= \cos^{-1} \frac{\hat{x} \cdot r_{xy}}{|r_{xy}| |\hat{x}|} \\ &= \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}} \end{aligned}$$

if $y < 0$, $\theta = -\theta$. Since we don't use regular procedure to convered cartesian coordinate to spherical coordinate due to $\tan^{-1} \frac{x}{y}$ returns $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, it produces more branch.

- ϕ is

$$\begin{aligned} \phi &= \cos^{-1} \frac{z}{r} \\ &= \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

- Finally, we get

if $y \geq 0$

$$(\theta, \phi, r) = \left(\cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}, \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}, 1 \right)$$

if $y < 0$

$$(\theta, \phi, r) = \left(-\cos^{-1} \frac{x}{\sqrt{x^2 + y^2}}, \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}, 1 \right)$$

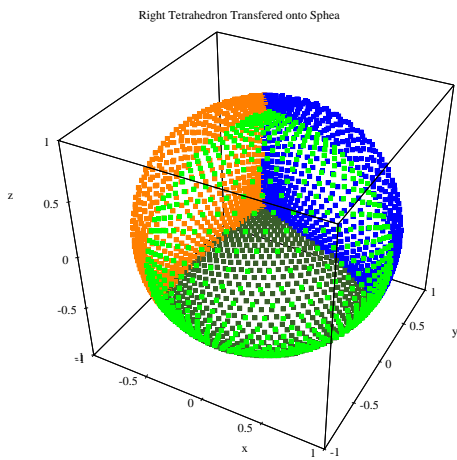
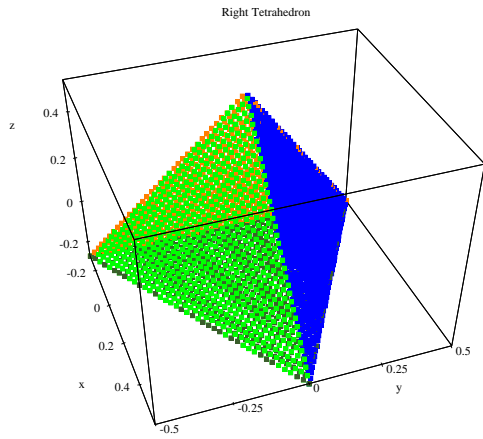
why $r = 1$ is that because we are making sphere with radius 1.

5 Cylindrical Coordinate Map

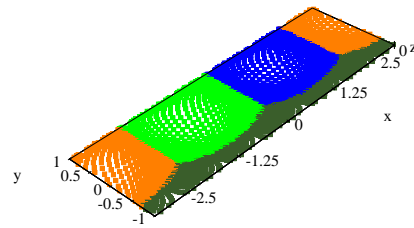
Take cylindrical projection of the sphere makes the world atlas map of the sphea. In this map, the vertical axis represents latitude and the horizontal axis represents longitude. So the conversion is here.

$$x_{map} = \theta$$

$$y_{map} = \sin \frac{\pi}{2} - \phi$$



Cylindrical Projection of Right Tetrahedron



□